Absolute continuity of the spectrum of a Landau Hamiltonian perturbed by a generic periodic potential

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ABSOLUTE CONTINUITY OF THE SPECTRUM OF A LANDAU HAMILTONIAN PERTURBED BY A GENERIC PERIODIC POTENTIAL

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Abstract. Consider Γ, a non-degenerate lattice in $\mathbb{R}^2$ and a constant magnetic field B with a flux though a cell of Γ that is a rational multiple of 2π. We prove that for a generic Γ-periodic potential V, the spectrum of the Landau Hamiltonian with magnetic field B and periodic potential V is purely absolutely continuous.

Résumé. On considère Γ, un réseau non-dégénéré dans $\mathbb{R}^2$ et un champ magnétique constant B dont le flux à travers une cellule du réseau est un multiple rationnel de 2π. On démontre que, pour un potentiel Γ-périodique V continu générique, le spectre du hamiltonien de Landau de champ magnétique constant B perturbé par le potentiel périodique V est purement absolument continu.

Written in the Coulomb gauge, on $L^2(\mathbb{R}^2)$, the Landau Hamiltonian is defined by

$$H = (-i\nabla - A)^2, \quad \text{where} \quad A(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \tag{1}$$

Let $\Gamma = \oplus_{i=1}^2 \mathbb{Z}e_i$ be a non-degenerate lattice such that

$$B\, e_1 \wedge e_2 \in 2\pi\mathbb{Q}. \tag{2}$$

Define the set of real valued, continuous, Γ-periodic functions

$$C_\Gamma = \{V \in C(\mathbb{R}^2, \mathbb{R});\ \forall x \in \mathbb{R}^2,\ \forall \gamma \in \Gamma,\ V(x+\gamma) = V(x)\}. \tag{3}$$

The space C_Γ is endowed with the uniform topology, the associated norm being denoted by $\|\cdot\|$.
Our main result is

Theorem 1. *There exists a G_δ-dense subset of C_Γ such that, for V in this set, the spectrum of $H(V) := H + V$ is purely absolutely continuous.*

The absence of singular continuous spectrum can be obtained from the sole analytic direct integral representation of $H(V)$ that we use below ([2]).
Our result is optimal in the sense that there are examples of periodic V for which the spectrum of H contains eigenvalues such as V constant. Of

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course, it is a natural question to wonder whether the constant potential is the only periodic one for which the spectrum exhibits eigenvalues.

The proof of Theorem 1 consists in several steps. We first reduce the problem via magnetic Floquet theory. Therefore, we introduce the magnetic translations [5]. For the two-dimensional, constant, transverse magnetic field problem, they are defined as follows. For any field strength $B \in \mathbb{R}$, any vector $\alpha \in \mathbb{R}^2$, and $f \in C_0^\infty(\mathbb{R}^2)$, we define the magnetic translation by α to be

$$U_\alpha^B f(x) := e^{\frac{iB}{2} x\wedge\alpha} f(x+\alpha) = e^{\frac{iB}{2}(x_1\alpha_2 - x_2\alpha_1)} f(x+\alpha). \tag{4}$$

For $(\alpha, \beta) \in (\mathbb{R}^2)^2$, we have the commutation relations

$$U_\alpha^B U_\beta^B = e^{iB\,\alpha\wedge\beta}\, U_\beta^B U_\alpha^B. \tag{5}$$

In a standard way, the family $\{U_\alpha^B;\ \alpha \in \mathbb{R}^2\}$ extends to a projective unitary representation of $\mathbb{R}^2$ on $L^2(\mathbb{R}^2)$. We note that

$$[U_\alpha^B, H] = 0 \quad \text{and} \quad [U_\alpha^B, V] = 0. \tag{6}$$

Let (e_1, e_2) be a "fundamental basis" of the lattice Γ i.e. $\Gamma = \oplus_{i=1}^2 \mathbb{Z} e_i$. For $j \in \{1, 2\}$, we define the unitary $U_j^B := U_{e_j}^B$ by (4). By assumption (2), one has

$$Be_1 \wedge e_2 = 2\pi p/q, \quad \text{for} \quad (p,q) \in \mathbb{Z}\times\mathbb{N},\ p\wedge q = 1. \tag{7}$$

It follows from (2) and (5) that the unitary operators $\{(U_1^B)^q, U_2^B\}$ satisfy the commutation relation

$$(U_1^B)^q U_2^B = e^{iqBe_1\wedge e_2} U_2^B (U_1^B)^q = e^{i2\pi p} U_2^B (U_1^B)^q = U_2^B (U_1^B)^q,$$

so the pair generates an Abelian group.
One checks that

$$[(U_1^B)^q, H(V)] = 0 = [U_2^B, H(V)] \tag{8}$$

Consider Γ', the sublattice of Γ defined by $\Gamma' = q\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Its dual lattice $(\Gamma')^*$ is given by

$$(\Gamma')^* = \{\gamma^* \in \mathbb{R}^2;\ \forall \gamma' \in \Gamma',\ \gamma^*\cdot\gamma' \in 2\pi\mathbb{Z}\}.$$

For any $\gamma' = q\gamma_1' e_1 + \gamma_2' e_2 \in \Gamma'$, define the phase $\Theta_q(\gamma')$ by

$$\Theta_q(\gamma') = e^{iBe_1\wedge e_2 q\gamma_1'\gamma_2'/2} = e^{i\pi p\gamma_1'\gamma_2'} \in \{-1, +1\}. \tag{9}$$

This allows us to define a unitary representation of the sublattice Γ' by

$$W_{q,\gamma'}^B = \Theta_q(\gamma') U_{\gamma'}^B. \tag{10}$$

It is easy to check that

$$W_{q,\gamma}^B W_{q,\gamma'}^B = W_{q,\gamma+\gamma'}^B, \quad \forall(\gamma,\gamma') \in (\Gamma')^2.$$

We define the transformation T^B on smooth functions by

$$(T^B f)(x,\theta) = \sum_{\gamma'\in\Gamma'} e^{i\theta\cdot(x+\gamma)} (W_{q,\gamma'}^B f)(x), \quad \theta \in (\mathbb{R}^2)^*/(\Gamma')^*.$$

Again, a simple calculation shows that

$$(W_{q,\gamma'}^B T^B f)(x,\theta) = (T^B f)(x,\theta).$$

We define a function space $\mathcal{H}_{B,p}$ by

$$\mathcal{H}_{B,p} = \{v \in L^2_{loc}(\mathbb{R}^2) \mid W^B_{q,\gamma'} v = v;\ \forall \gamma' \in \Gamma'\}.$$

It then follows that T^B extends to a unitary map

$$T^B :\ L^2(\mathbb{R}^2) \to L^2((\mathbb{R}^2)^*/(\Gamma')^*), \mathcal{H}_{B,p}).$$

Given this structure, it is clear that the Hamiltonian H admits a direct integral decomposition (see e.g. [4]) over $(\mathbb{R}^2)^*/(\Gamma')^*$, so that

$$T^B H(V)(T^B)^* = \int^{\oplus}_{(\mathbb{R}^2)^*/(\Gamma')^*} H(\theta, V)\, d\theta.$$

The operator $H(\theta, V)$ is self-adjoint on the Sobolev space $\mathcal{H}^2_{B,p}$, the local Sobolev space of order two of functions in $\mathcal{H}_{B,p}$ and one computes

$$H(\theta, V) = (i\nabla + A - \theta)^2 + V. \tag{11}$$

This operator has a compact resolvent. Consequently, the spectrum is discrete and consists of eigenvalues of finite multiplicity, say, $(E_j(V,\theta))_{j\in\{1,2,\dots\}}$ labeled in increasing order and repeated according to multiplicity. For $n \geq 1$, the function $(\theta, V) \in (\mathbb{R}^2)^*/(\Gamma')^* \times C_\Gamma \mapsto E_n(V,\theta)$ is locally uniformly Lipschitz continuous; this follows from the variational principle (see e.g. [4]) and the fact that $(i\nabla - A)$ is H-bounded with relative bound 0. We endow the space $(\mathbb{R}^2)^*/(\Gamma')^* \times C_\Gamma$ with the norm $\|(\theta, V)\| = |\theta| + \|V\|$.

It is well known (see [4, 7]) that Theorem 1 is a corollary of

Theorem 2. *There exists a G_δ-dense subset of C_Γ such that, for V in this set, none of the functions $\theta \mapsto E_n(V, \theta)$, $n \geq 1$, is constant.*

Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ and $V \in C_\Gamma$. Let $n \geq 1$.

Definition 1. $E_n(\theta_0, V_0)$ is an analytically degenerate eigenvalue of $H(\theta_0, V_0)$ if and only if there exists $\delta > 0$ and an orthonormal system of p functions, say $(\theta, V) \mapsto \varphi_j(\cdot, \theta, V)$, $j \in \{1, \cdots, p\}$, defined and real analytic on $U_{\theta_0,V_0} := \{\|(\theta, V) - (\theta_0, V_0)\| < \delta\}$ valued in $\mathcal{H}^2_{B,p}$ such that, for all $(\theta, V) \in U_{\theta_0,V_0}$,

- the functions $(\varphi_j(\cdot, \theta, V))_{1\leq j\leq p}$ span the kernel of $H(\theta, V) - E_n(\theta, V)$,
- one has

$$H(\theta, V)\varphi_j(\theta, V) = E_n(\theta, V)\varphi_j(\theta, V) \quad \text{for} \quad 1 \leq j \leq p.$$

Remark 1. As one can see from the proof of Lemma 2, to say that $E_n(\theta, V)$ is analytically degenerate near (E_0, V_0) is equivalent to say that the multiplicity of $E_n(\theta, V)$ is constant in some neighborhood of (E_0, V_0).

Theorem 2 is a consequence of the following two lemmas

Lemma 1. *Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ and $V_0 \in C_\Gamma$ such that V_0 is not a constant. Assume that $E_n(\theta_0, V_0)$ is an analytically degenerate eigenvalue of $H(\theta_0, V_0)$. Then, for any $\varepsilon > 0$, there exists $V \in \{\|V - V_0\| < \varepsilon\}$ such that $\theta \mapsto E_n(\theta, V)$ is not constant.*

and

Lemma 2. *Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ and $V_0 \in C_\Gamma$. Fix $n \geq 1$. Then, for any $\varepsilon > 0$, there exists $(\theta_\varepsilon, V_\varepsilon) \in \{\|(\theta, V) - (\theta_0, V_0)\| < \varepsilon\}$ and $\delta > 0$ such that $E_n(\theta, V)$ is an analytically degenerate eigenvalue of $H(\theta, V)$ for $(\theta, V) \in \{\|(\theta, V) - (\theta_\varepsilon, V_\varepsilon)\| < \delta\}$.*

Remark 2. In general, in Lemma 2, the multiplicity of the eigenvalue is one.

How to complete the proof of Theorem 2 using Lemmas 1 and 2 is straightforward. For any $n \geq 1$, the set of V in C_Γ such that $\theta \mapsto E_n(\theta, V)$ is not constant is open (as the Floquet eigenvalues are locally uniformly Lipschitz continuous in $(\theta; V)$). In view of Lemma 1 and 2, for any $n \geq 1$, the set of V in C_Γ such that $\theta \mapsto E_n(\theta, V)$ is not constant is dense. Hence, the set of V where none of $(\theta \mapsto E_n(\theta, V))_{n\geq 1}$ is constant is a countable intersection of dense open sets i.e. a G_δ-dense set. This completes the proof of Theorem 2.

The proof of Lemma 2. Fix $\varepsilon > 0$. Pick $V_0 \in C_\Gamma$ and $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$. Then, $E_n(V_0, \theta_0)$ is an isolated eigenvalue of $H(\theta_0, V_0)$ of multiplicity say $N_0 = N(\theta_0, V_0)$. Let $\delta > 0$ be such that $E_n(V_0, \theta_0)$ be the only eigenvalue of $H(\theta_0, V_0)$ in $D(E_n(V_0, \theta_0), 2\delta)$, the disk of center $E_n(V_0, \theta_0)$ and radius 2δ. The projector onto the eigenspace associated to $E_n(V_0, \theta_0)$ and $H(\theta_0, V_0)$ is given by Riesz's formula

$$\Pi(\theta_0, V_0) = \frac{1}{2i\pi} \int_{|z - E_n(V_0,\theta_0)|=\delta} (z - H(\theta_0, V_0))^{-1} dz.$$

It is well known (see e.g. [4, 3]) that, there exists $\varepsilon_0 \in (0, \varepsilon)$ such that, for $\|(\theta, V) - (\theta_0, V_0)\| < \varepsilon_0$, the projector onto the eigenspace associated to the spectrum of $H(\theta, V)$ in $D(E_n(V_0, \theta_0), \delta)$ is given by

$$\Pi(\theta, V) = \frac{1}{2i\pi} \int_{|z - E_n(V_0,\theta_0)|=\delta} (z - H(\theta, V))^{-1} dz. \tag{12}$$

In particular, the rank of this projector is constant and equal to N_0, the multiplicity of $E_n(V_0, \theta_0)$ as an eigenvalue of $H(\theta_0, V_0)$.
Consider the operator $M(\theta, V) = \Pi(\theta, V)H(\theta, V)$. Its eigenvalues are the eigenvalues of $H(\theta, V)$ in $D(E_n(\theta_0, V_0), \delta)$ and it has finite rank N_0. Let $(\psi_j)_{1\leq j\leq N_0}$ be an orthonormal basis of eigenvectors of $H(\theta_0, V_0)$ associated to the eigenvalue $E_n(\theta_0, V_0)$. For $j \in \{1, \cdots, N_0\}$, set $\psi_j(\theta, V) = \Pi(\theta, V)\psi_j$ and let $G(\theta, V)$ be the Gram matrix of these vectors. Then,

$$G(\theta, V) - \mathrm{Id}_{N_0} = O(\|(\theta, V) - (\theta_0, V_0)\|)$$

and the vectors

$$\begin{pmatrix} \varphi_1(\theta, V) & \cdots & \varphi_{N_0}(\theta, V) \end{pmatrix} = \begin{pmatrix} \psi_1(\theta, V) & \cdots & \psi_{N_0}(\theta, V) \end{pmatrix} \sqrt{G^{-1}(\theta, V)}$$

form an orthonormal basis of $\Pi(\theta, V)\mathcal{H}_{B,p}$.
E is an eigenvalue of $H(\theta, V)$ in $D(E_n(V_0, \theta_0), \delta)$ if and only if

$$P(E; \theta, V) = \mathrm{Det}\,(\tilde{M}(\theta, V) - E) = 0$$

where Det denotes the determinant and $\tilde{M}(\theta, V)$, the matrix of $M(\theta, V)$ in the basis $(\varphi_1(\theta, V), \cdots, \varphi_{N_0}(\theta, V))$.
Then, either of two things occur:

(1) there exists $\varepsilon > 0$ and a function $(\theta, V) \mapsto E(\theta, V)$ such that, for $\|(\theta, V) - (\theta_0, V_0)\| < \varepsilon$, one has

$$P(E(\theta,V),\theta,V) = \partial_E P(E(\theta,V),\theta,V) = \cdots = \partial_E^{N_0-1} P(E(\theta,V),\theta,V) = 0$$

in which case, one has

$$P(E,\theta,V) = (E - E(\theta,V))^{N_0}.$$

So $E_n(\theta, V)$ is the only eigenvalue of the matrix $\tilde{M}(\theta, V)$. For θ and V real, $\tilde{M}(\theta, V)$ is Hermitian hence it is equal to $E_n(\theta, V)\,\mathrm{Id}_{N_0}$.
Pick now V complex such that $\|(\theta, V) - (\theta_0, V_0)\| < \varepsilon/4$. We can write $V = V_r + iV_i$ with both $V_r \in \mathcal{C}_\Gamma$ and $V_i \in \mathcal{C}_\Gamma$. For $z \in D(0,2)$, $\|(\theta, V_r + zV_i) - (\theta_0, V_0)\| < \varepsilon$. Hence, $z \mapsto \tilde{M}(\theta, V_r + zV_i)$ and $z \mapsto E_n(\theta, V_r + zV_i)$ are analytic. Above, we have proved that, for z real, one has

$$\tilde{M}(\theta, V_r + zV_i) = E_n(\theta, V_r + zV_i)\mathrm{Id}_{N_0}.$$

By analytic continuation, this stays true for z in $D(0,2)$ in particular for $z = i$ i.e. $\tilde{M}(\theta, V)$ is Hermitian hence it is equal to $E_n(\theta, V)$ times the identity.
So, $(\theta, V) \mapsto E(\theta, V)$ in an analytically degenerate eigenvalue of $H(\theta, V)$ (of order N_0).

Remark 3. Actually, using the normal Jordan form for matrices instead of the Hermitian nature of the matrix, we only need to know that $\tilde{M}(\theta_0, V_0)$ is reducible to conclude that if the multiplicity of $E_n(\theta, V)$ is constant then the eigenvalue is analytically degenerate.

(2) or, for any $\varepsilon > 0$, there exists $N_1 < N_0$ and (θ_1, V_1) such that $|(\theta_1, V_1) - (\theta_0, V_0)| < \varepsilon$ and

$$\begin{aligned} P(E(\theta_1,V_1),\theta_1,V_1) &= \partial_E P(E(\theta_1,V_1),\theta_1,V_1) \\ &= \cdots = \partial_E^{N_1-1} P(E(\theta_1,V_1),\theta_1,V_1) = 0 \end{aligned}$$

and

$$\partial_E^{N_1} P(E(\theta_1,V_1),\theta_1,V_1) \neq 0.;$$

in this case, $E(\theta_1, V_1)$ is an eigenvalue of multiplicity $N_1 \leq N_0 - 1$ of $H(\theta_1, V_1)$.

In the first case, Lemma 2 is proven. In the second case, we can then start the process over again near (θ_1, V_1). After at most N_0 such reductions we will have constructed the pair $(\theta_\varepsilon, V_\varepsilon)$ announced in Lemma 2. This completes the proof of Lemma 2. □

We now turn to the proof of Lemma 1.

Proof of Lemma 1. Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ and $V_0 \in C_\Gamma$. Assume that $E_n(\theta_0, V_0)$ is an analytically degenerate eigenvalue of $H(\theta_0, V_0)$. Let us write $E(\theta, V) := E_n(\theta, V)$. Assume that the conclusions of Lemma 1 is false. Then, there exists $\varepsilon > 0$ such that for any $V \in \mathcal{C}_\Gamma$ such that $\|V - V_0\| \leq \varepsilon$, the function $\theta \mapsto E(\theta, V)$ is constant. In particular, we can slightly change V_0 to assume that it is real analytic and the same conclusion still holds.
Pick $U \in \mathcal{C}_\Gamma$ such that $\|U\| = 1$ and set $V_t = V_0 + tU$, t complex small. As $E(\theta_0, V_0)$ is an analytically degenerate eigenvalue of $H(\theta_0, V_0)$, there exists

$\varepsilon > 0$ and $\varphi(\theta, t)$ real analytic in (θ, t) such that, for $|t| \leq \varepsilon$ and $|\theta - \theta_0| \leq \varepsilon$, one has

$$(H(\theta, t) - E(\theta, t))\varphi(\theta, t) = 0, \quad \|\varphi(\theta, t)\| = 1. \tag{13}$$

Moreover $(\theta, t) \mapsto E(\theta, t)$ is real analytic.
Differentiating the eigenvalue equation (13) for φ in t yields

$$(H(\theta, t) - E(\theta, t))\partial_t\varphi(\theta, t) = [\partial_t E(\theta, t) - U]\varphi(\theta, t). \tag{14}$$

We note that, for $\gamma' \in \Gamma'$,

$$W^B_{\gamma'}(\partial_t\varphi(\theta, t)) = \partial_t W^B_{\gamma'}(\varphi(\theta, t)) = \partial_t\varphi(\theta, t)$$

so $\partial_t\varphi(\theta, t) \in \mathcal{H}_{B,p}$.
Using (13) and the self-adjointness of $H(\theta, t)$ on $\mathcal{H}_{B,p}$, one obtains

$$\partial_t E(\theta, t) = \langle U\varphi(\theta, t), \varphi(\theta, t)\rangle. \tag{15}$$

We now assume that $E(\theta, t)$ does not depend on θ in some neighborhood of θ_0 and for t small i.e.

$$\nabla_\theta E(\theta, t) = 0.$$

So differentiating (15) with respect to θ, we obtain that

$$\begin{aligned} 0 &= \partial_t\nabla_\theta E(\theta, t) = \nabla_\theta\partial_t E(\theta, t) \\ &= \langle U\varphi(\theta, t), \nabla_\theta\varphi(\theta, t)\rangle + \langle U\nabla_\theta\varphi(\theta, t), \varphi(\theta, t)\rangle \\ &= 2\mathrm{Re}\left[\langle U\varphi_k(\theta, t), \nabla_\theta\varphi_k(\theta, t)\rangle\right] \end{aligned}$$

Here, the real part is meant coordinate wise.
At $t = 0$, we then get that

$$\begin{aligned} 0 &= \mathrm{Re}\left[\langle U\varphi(\theta, 0), \nabla_\theta\varphi_k(\theta, 0)\rangle\right] \\ &= \int_{\mathbb{R}^2/\Gamma} U(x)\,\mathrm{Re}\left(\nabla_\theta\varphi_k(x;\theta, 0)\overline{\varphi_k(x;\theta, 0)}\right)dx. \end{aligned} \tag{16}$$

So, if for all $U \in \mathcal{C}_\Gamma$ such $\|U\| = 1$ and for t small, we know that $\theta \mapsto E(\theta, t)$ is constant in some neighborhood of θ_0, we obtain that (16) holds for all $U \in \mathcal{C}_\Gamma$ such that $\|U\| = 1$. So, for θ near θ_0, one has

$$\forall x \in \mathbb{R}^2, \quad 2\mathrm{Re}(\nabla_\theta\varphi(x;\theta, 0)\overline{\varphi(x;\theta, 0)}) = \nabla_\theta\left(|\varphi(x;\theta, 0)|^2\right) \equiv 0. \tag{17}$$

The operator $(i\nabla - A - \theta)^2 + V_0$ being elliptic with real analytic coefficients, it is analytically hypoelliptic (see, e.g. [6]); hence, $x \mapsto \varphi(x;\theta, 0)$ is real analytic on $\mathbb{R}^2$. For $|\theta - \theta_0| \leq \varepsilon$, let $O_\theta \subset \mathbb{R}^2$ be the open set where the function $x \mapsto \varphi(x;\theta, 0)$ does not vanish. By (17), this set is independent of θ; we denote it by O. As $\varphi(\theta, 0) \in \mathcal{H}_{B,p}$, O is invariant by the translations by a vector in Γ'. Define Z by $Z := \mathbb{R}^2 \setminus O$. Z is also Γ'-periodic. Let C be the fundamental cell of the lattice Γ'. As Z is the set of zeros of the real analytic function $x \mapsto \varphi(x;\theta_0, 0)$ and as $Z \cap C$ is compact, we know that $Z \cap C$ has the following finite decomposition (see e.g. [1])

$$Z \cap C = \bigcup_{p=1}^{p_0} \mathcal{A}_p \tag{18}$$

where the union is disjoint and, for $1 \leq p \leq p_0$, one has

(1) the set $\mathcal{A}_p$ either is reduced to a single point or is a connected real-analytic curve (i.e. a connected real analytic manifold of dimension 1);
(2) if $p \neq p'$ and $\mathcal{A}_p \cap \overline{\mathcal{A}_{p'}} \neq \emptyset$, then
   - $\mathcal{A}_p \subset \overline{\mathcal{A}_{p'}}$,
   - $\mathcal{A}_p$ is reduced to a single point,
   - $\mathcal{A}_{p'}$ is a real-analytic curve;

(3) assume $\mathcal{A}_p = \{x_0\}$. Then, either x_0 is isolated in $Z \cap C$ or, for some $\varepsilon_0 > 0$ sufficiently small, one has
$$Z \cap C \cap \dot{\overline{D}}(x_0, \varepsilon_0) = \bigcup_{p' \in E} \mathcal{A}_{p'} \cap \dot{\overline{D}}(x_0, \varepsilon_0)$$
where E is a non empty, finite set of indices such that, for $p' \in E$, the set $\mathcal{A}_{p'}$ is a real analytic curve.
Here, $\dot{\overline{D}}(x_0, \varepsilon_0) = \{0 < |x - x_0| \leq \varepsilon_0\}$.

Let $Z_0 = \cup_{\#\mathcal{A}_p = 1} \mathcal{A}_p$ be the set of the points composing the point components in the above decomposition.

Remark 4. As our Hamiltonian has no real symmetry i.e. the partial differential operator does not have real coefficients and as we are working in two space dimensions, it is reasonable to expect that the nodal set of an eigenfunction, if it is no empty, is actually made of points.

We will use the following

Lemma 3. *Let Z_∇ be the set of points x_0 in C such that $\varphi(x_0; \theta, 0) = 0$ and $\nabla\varphi(x_0; \theta, 0) = 0$. Then, Z_∇ consists of isolated points.*

We postpone the proof of Lemma 3 to complete that of Lemma 1.
Consider a horizontal straight line $L_x = x + \mathbb{R} \times \{0\}$ that does not intersect $Z_0 \cup Z_\nabla$. As the other components of Z are real analytic curves, possibly shifting this line, we can assume that it intersects these curves transversally in finitely many points. For $\delta > 0$, define the strip S_x^δ by
$$S_x^\delta = x + \mathbb{R} \times (-\delta, \delta).$$
Then, there exists $\delta > 0$ such that

- $\overline{S_x^\delta} \cap (Z_0 \cup Z_\nabla) = \emptyset$,
- S_x^δ intersects Z in C at, at most, finitely many vertical curves, and these curves partition the strip in a finite number of open domains (see Fig. 1). Here, vertical means that the curves can be parametrized by the coordinate x_1.

Recall that Z is Γ' periodic. Hence, we get that
$$S_x^\delta \setminus Z = \bigcup_{\gamma' \in q\mathbb{Z}e1} \bigcup_{k=1}^{s} \gamma' + D_k \quad \text{and} \quad Z \cap S_x = \bigcup_{\gamma' \in q\mathbb{Z}e1} \bigcup_{k=1}^{s} \gamma' + C_k$$
where, to fix ideas we assume that C_k is the left boundary of D_k.
We prove

Lemma 4. *Let D be one of the domains $\gamma' + D_k$ for some $1 \leq k \leq s$ and some $\gamma' \in q\mathbb{Z}e_1$.*

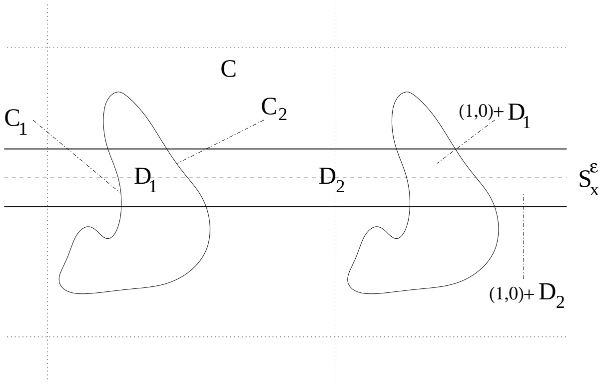

Figure 1: The strip

For θ such that $|\theta-\theta_0|<\varepsilon$, there exists two real continuous $x\in\overline{D}\mapsto g_D(x;\theta)\in\mathbb{R}$ and $x\in\overline{D}\mapsto\psi_D(x)\in\mathbb{R}^+$, such that

$$\forall x\in D,\quad \varphi(x;\theta,0)=e^{ig_D(x;\theta)}\psi_D(x). \tag{19}$$

and such that

- *for any $x_0\in D$, $(x,\theta')\mapsto g_D(x;\theta')$ is real analytic in a neighborhood of (x_0,θ),*
- *let D' be another domain in the collection $(\gamma'+D_k)_{\gamma',k}$; if $\overline{D}\cap\overline{D'}\neq\emptyset$ and D' is to the left of D, then, for $x\in\overline{D}\cap\overline{D'}$, one has*

$$g_D(x;\theta)=g_{D'}(x;\theta)+\pi. \tag{20}$$

Before turning to the proof of this result, let us complete the proof of Lemma 1.
Recall that $\varphi(\theta,0)\in\mathcal{H}_{B,p}$ i.e. that $W^B_{q,\gamma'}\varphi(\theta,0)=\varphi(\theta,0)$ for all $\gamma'\in\Gamma'$. By (10), the definition of $W^B_{q,\gamma'}$, the functions coming into the decomposition given in Lemma 4 must satisfy, for $\gamma'\in q\mathbb{Z}e_1$ and $x\in D_k$

$$g_{\gamma'+D_k}(x+\gamma',\theta)=g_{D_k}(x,\theta)-\frac{B}{2}x\wedge\gamma'-\pi\gamma'_1\gamma'_2 \tag{21}$$

and

$$\psi_{\gamma'+D_k}(x+\gamma')=\psi_{D_k}(x).$$

For D, one of the domains $(\gamma'+D_k)_{\gamma',k}$, plug the representation (19) into the eigenvalue equation (13) to obtain that, on D, one has

$$(i\nabla_x-A-\theta-\nabla_x g_D)^2\psi_D+V_0\psi_D=E\psi_D$$

where $E=E(\theta,0)$ as it does not depend on θ. As V_0, ψ and g real valued, we can take the complex conjugate of this equation to obtain that, on O, one has

$$(i\nabla_x+A+\theta+\nabla_x g_D)^2\psi_D+V_0\psi_D=E\psi_D.$$

Summing the last two equations, one finally obtains that, on D, one has

$$(A+\theta+\nabla_x g_D)^2\psi_D=(E-V_0)\psi_D+\Delta\psi_D$$

As A, ψ_D, E and V_0 do not depend on θ, as ψ_D does not vanish on D, this equation implies that, for $x\in D$, the function $\theta\mapsto\theta+\nabla_x g_D(x,\theta)$ does not depend on θ. Hence, there exists a function $x \in \overline{D} \mapsto h_D(x)$ that is real analytic in D and a real analytic function $\theta \mapsto c_D(\theta)$ such that, for $|\theta - \theta_0| \leq \varepsilon$ and $x \in D$, one has

$$g_D(x, \theta) = -\theta \cdot x + h_D(x) + c_D(\theta). \tag{22}$$

We note that (20) in Lemma 4 tells us that, if D' is to the left of D and $\overline{D'} \cap \overline{D} \neq \emptyset$, then we may choose

$$c_D(\theta) = c_{D'}(\theta) + \pi. \tag{23}$$

We now plug the representation (22) into (21) and use (23) to obtain that, for $|\theta - \theta_0| \leq \varepsilon$, $\gamma' = \gamma_1' e_1 \in p\mathbb{Z}e_1$ and $x \in D$,

$$\theta \cdot \gamma' = h_{\gamma'+D}(x) - h_D(x) + \frac{B}{2} x \wedge \gamma' + \pi \gamma_1' \gamma_2' - s\gamma_1' \pi.$$

This is absurd as the left hand side of this expression depends on θ and the right does not.
This completes the proof of Lemma 1. □

We now turn to the proof of Lemmas 3 and 4.

Proof of Lemma 3. First, the set $Z_\nabla \cap C$ is real analytic so can be decomposed in the same way as $Z \cap C$. If it does not consist of isolated points, then it contains an analytic curve, say, c. Pick a point x^0 in this curve. Near $x^0 = (x_1^0, x_2^0)$ assume, without restriction, that the curve is parametrized by $x_2 = c(x_1)$ where c is real analytic.
Define the functions $u(x) = \mathrm{Re}(\varphi(x; \theta, 0))$ and $v(x) = \mathrm{Im}(\varphi(x; \theta, 0))$. They are real analytic, real valued and satisfy

- as $\varphi(\theta, 0)$ is a solution to the eigenvalue equation (13),

$$\begin{aligned} (-\Delta u) + (A - \theta)^2 u + 2A \cdot \nabla v &= (E - V)u, \\ (-\Delta v) + (A - \theta)^2 v - 2A \cdot \nabla u &= (E - V)v; \end{aligned} \tag{24}$$

  here, we used $\mathrm{div} A = 0$;
- on c, one has

$$0 = u = v = \partial_1 u = \partial_1 v = \partial_2 u = \partial_2 v \tag{25}$$

  by the definition of Z_∇.

Let us prove inductively that, for any $\alpha \in \mathbb{N}^2$, $\partial^\alpha u = \partial^\alpha v = 0$ on c. Assume that, for $\alpha_1 + \alpha_2 \leq N$, one has $\partial_1^{\alpha_1} \partial_2^{\alpha_2} u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} v = 0$. Let us prove that it still holds for $\alpha_1 + \alpha_2 = N + 1$.
For $\alpha_1 + \alpha_2 \leq N + 1$, differentiating $\alpha_1 - 1$ times equations (24) in x_1 and $\alpha_2 - 1$ times in x_2 yields that, on c, one has

$$\begin{aligned} \partial_1^{\alpha_1+1} \partial_2^{\alpha_2-1} u + \partial_1^{\alpha_1-1} \partial_2^{\alpha_2+1} u &= \sum_{\beta_1+\beta_2 \leq N} a_{\beta_1\beta_2} \partial^\beta u + b_{\beta_1\beta_2} \partial^\beta v = 0, \\ \partial_1^{\alpha_1+1} \partial_2^{\alpha_2-1} v + \partial_1^{\alpha_1-1} \partial_2^{\alpha_2+1} v &= \sum_{\beta_1+\beta_2 \leq N} c_{\beta_1\beta_2} \partial^\beta u + d_{\beta_1\beta_2} \partial^\beta v = 0. \end{aligned} \tag{26}$$

Differentiating $\partial_1^{\alpha_1} \partial_2^{\alpha_2} u = 0$ along c, we get

$$\left(\partial_1^{\alpha_1+1} \partial_2^{\alpha_2} u\right)(x_1, c(x_1)) + c'(x_1) \left(\partial_1^{\alpha_1} \partial_2^{\alpha_2+1} u\right)(x_1, c(x_1)) = 0 \tag{27}$$

Using this for $(\alpha_1, \alpha_2) = (N, 0)$ and $(\alpha_1, \alpha_2) = (N-1, 1)$ and the first equation in (26) for $(\alpha_1, \alpha_2) = (N, 1)$, we get the system

$$\begin{cases} \partial_1^{N+1} u + c' \partial_1^N \partial_2 u & = 0 \\ \partial_1^N \partial_2 u + c' \partial_1^{N-1} \partial_2^2 u & = 0 \\ \partial_1^{N+1} u + c' \partial_1^{N-1} \partial_2^2 u & = 0 \end{cases}$$

which implies that

$$\partial_1^{N+1} u = \partial_1^N \partial_2 u = \partial_1^{N-1} \partial_2^2 u = 0.$$

Let us assume that $c'(x_1) \neq 0$. Then, using (26) inductively, we get that $\partial_1^{N+1-\alpha} \partial_2^\alpha u = 0$ for all $0 \leq \alpha \leq N+1$.
If c' does not vanish on the whole curve, we just work near a point where it does not vanish. If c' vanishes on the whole curve, then the curve is a straight horizontal line, say, $x_2 = 0$ and we proceed as follows. By differentiation of (25), we immediately get that, on c, one has

$$\partial_1^{N+1} u = \partial_1^N \partial_2 u = \partial_1^{N+1} v = \partial_1^N \partial_2 v = 0$$

Then, (26) and the induction assumption yield, for $0 \leq \alpha \leq N$,

$$0 = -\partial_1^{N+1-\alpha} \partial_2^\alpha u = \partial_1^{N-\alpha-1} \partial_2^{\alpha+2} u,$$
$$0 = -\partial_1^{N+1-\alpha} \partial_2^\alpha v = \partial_1^{N-\alpha-1} \partial_2^{\alpha+2} v.$$

Finally we proved that, if $Z_\nabla \cap C$ contains a curve, the functions $(\partial_x^\alpha)\varphi(\theta, 0)$ vanish identically on this curve. As $\varphi(\theta, 0)$ is real analytic, this implies that this function vanishes identically which contradicts the assumption that its norm in $\mathcal{H}_{B,p}$ is 1.
This completes the proof of Lemma 3. □

Proof of Lemma 4. Clearly, in the domains $(D_k)_{\leq k \leq s}$ and their translates, the decomposition (19) is the decomposition into argument and modulus of the complex number $\varphi(x; \theta, 0)$. As $\varphi(x; \theta, 0)$ does not vanish, its argument and modulus are also real analytic. So we only need to study what happens at the crossing of one of the curves $(C_k)_{\leq k \leq s}$. So, we study $x \mapsto \varphi(x; \theta, 0)$ near $x^0 \in C_k$.
As $S_x^\delta \cap (Z_0 \cup Z_\nabla) = \emptyset$, we know that $\nabla \varphi(x^0, \theta, 0) \neq 0$. Using the notation of the proof of Lemma 3 i.e. $u(x) = \mathrm{Re}(\varphi(x; \theta, 0)$ and $v(x) = \mathrm{Im}(\varphi(x; \theta, 0)$, we may assume that $\nabla u(x^0) \neq 0$. As the curve C_k is vertical, we know that $\partial_1 u(x^0) \neq 0$. We can then find a real analytic change of variables that maps a neighborhood of x^0 into a neighborhood of 0 and that maps the set $\{x;\ u(x) = 0\}$ into the straight line $\{x_1 = 0\}$. We perform this change of variables on u and v and call the function thus obtained again u and v. Then, in a neighborhood of 0, one has that

$$u(x_1, x_2) = 0 \Leftrightarrow x_1 = 0, \quad \partial_1 u(0,0) \neq 0, \quad v(0, x_2) = 0. \tag{28}$$

The functions u and v being real analytic, we can write them as

$$u(x_1, x_2) = \tilde{w}(x_2) + x_1 w(x_1, x_2) \quad \text{and} \quad v(x_1, x_2) = \tilde{t}(x_2) + x_1 t(x_1, x_2)$$

where all the functions are real analytic.
Then, (28) implies then that

$$w(0,0) \neq 0, \quad \tilde{w}(x_2) = \tilde{t}(x_2) = 0 \text{ identically}.$$

Hence, we obtain that

$$(u+iv)(x_1,x_2) = x_1(w+it)(x_1,x_2) \text{ where } |(w+it)(0,0)| \neq 0.$$

Changing back to the initial variables, if $x_1 \mapsto c(x_1)$ is a parametrization of the curve C_k near x^0, we see that, in U, a neighborhood of x^0, we can write

$$\varphi(x;\theta,0) = (x_2 - c(x_1))\psi(x) \text{ where } \psi(x^0) \neq 0.$$

Hence, for $x \in D_k \cap U$, one has

$$e^{ig_{D_k}(x;\theta)}\psi_{D_k}(x) = (x_2 - c(x_1))\psi(x), \quad x_2 \geq c(x_1)$$

and for $x \in D_{k-1} \cap U$, one has

$$e^{ig_{D_{k-1}}(x;\theta)}\psi_{D_{k-1}}(x) = (x_2 - c(x_1))\psi(x) = -(c(x_1) - x_2)\psi(x), \quad x_2 \leq c(x_1).$$

This implies that we can continue $g_{D_{k-1}}$ and g_{D_k} continuously up to the boundary C_k and that they satisfy the relation (20) on C_k.
This completes the proof of Lemma 4. □

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